ONE FORM OF SOLUTION OF THREE-DIMENSIONAL AXISYMMETRIC PROBLEMS OF ELASTICITY THEORY BY MEANS OF FUNCTIONS OF A COMPLEX VARIABLE AND THE SOLUTION OF THESE PROBLEMS FOR THE SPHERE

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A method of solution of three-dimensional axisymmetric problems by means of functions of a complex variable was proposed in [1-3]; there the equations of the problem were obtained by a rotation of the plane state about an axis of symmetry or by a linear translation of the axisymmetric state. Below (Section 1) the equations are obtained by means of a general solution [2-3] of the three-dimensional problem of elasticity theory in the form due to P.F. Papkovich. These equations are utilized for the solution of the first and second basic problems of elasticity theory for a sphere and for a region with a spherical cavity (Section 2).

1. General equations for axisymmetric problems. 1. In the case of axisymmetric deformation of a body of revolution, the components of elastic displacement w, u may be represented in the following form (cf., for example, [4]):

$$2Gw = 4(1-\nu) B_z - \frac{\partial}{\partial z}(zB_z + rB_r + B_0)$$

$$2Gu = 4(1-\nu) B_r - \frac{\partial}{\partial r}(zB_z + rB_r + B_0) \qquad (1.1)$$

Here ν is Poisson's ratio, G is the shear modulus, and B_z, B_r, B_0 are functions of the variables z, r satisfying the equations

$$\triangle B_z = 0, \qquad \triangle (B_r e^{i\theta}) = 0, \qquad \triangle B_0 = 0 \qquad (1.2)$$

$$\left(\triangle = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) .$$

In these equations and below z, r, θ are cylindrical coordinates, z being the axis of revolution.

We pass a cutting plane through the z-axis. In this section a symmetrical plane figure is obtained (Fig. 1). We consider symmetrically situated points t and $\bar{t}$. We introduce functions ϕ_1, ϕ_2, ϕ_3 of a complex variable with the aid of the equations

$$B_z(z, r) = \frac{1}{\pi i}\int_{\bar{t}}^{t} \varphi_1(\zeta)\,\frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}}$$

$$B_r(z, r) = \frac{1}{\pi i r}\int_{\bar{t}}^{t} \varphi_2(\zeta)\,\frac{(\zeta - z)\,d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}} \qquad (1.3)$$

$$B_0(z, r) = \frac{1}{\pi i}\int_{\bar{t}}^{t} \varphi_3(\zeta)\,\frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}}$$

$$(t = z + ri,\ \bar{t} = z - ri)$$

Here ζ is a complex variable in that same plane which varies from $\zeta = \bar{t}$ to $\zeta = t$. One of the branches of the function $\sqrt{(\zeta - t)(\zeta - t)}$ should be used in these equations; for the sake of definiteness we assume that

$$\sqrt{(\zeta - t)(\zeta - \bar{t})} = \sqrt{\rho_1\rho_2}\,e^{i(\omega_1+\omega_2)/2}$$

if

$$\zeta - t = \rho_1 e^{i\omega_1}, \qquad \zeta - \bar{t} = \rho_2 e^{i\omega_2}$$

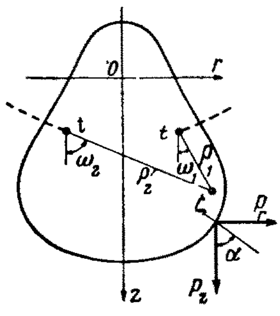

Fig. 1.

We will extend the branch cut (shown dotted in Fig. 1) to infinity.

We require that the values of the integrals in (1.3) be independent of the path of integration, if the integration is carried out along a smooth or piecewise smooth curve, lying wholly within the region and not crossing the branch cut. Moreover, we note that the left-hand sides of the equalities (1.3) must be real. Hence it follows that the functions $\phi_n(\zeta)$ must be holomorphic within the region considered, and

$$\operatorname{Re}\varphi_n(\zeta) = \operatorname{Re}\varphi_n(\zeta), \qquad \operatorname{Im}\varphi_n(\zeta) = -\operatorname{Im}\varphi_n(\bar{\zeta}) \qquad (n = 1, 2, 3) \qquad (1.4)$$

It is not difficult to convince oneself that the integrals in Formulas (1.3) converge absolutely and uniformly for $r > 0$. When $r = 0$ (i.e.

on the axis of symmetry), we have the equalities

$$B_z(z, 0) = \varphi_1(z), \qquad B_r(z, 0) = 0, \qquad B_0(z, 0) = \varphi_3(z) \tag{1.5}$$

Under these conditions the right-hand sides of (1.3) are continuous and differentiable functions of z and r. By direct calculation it may be shown that Equations (1.2) are identically satisfied.

Substituting (1.3) into (1.1), we obtain for the displacements

$$2Gw = \frac{1}{\pi i}\int_{\bar{t}}^{t} [\varkappa\varphi_1(\zeta) - z\varphi_1'(\zeta) - (\zeta - z)\varphi_2'(\zeta) - \varphi_3'(\zeta)] \frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}} \tag{1.6}$$

$$2Gu = \frac{1}{\pi i r}\int_{\bar{t}}^{t} [\varkappa\varphi_2(\zeta) - z\varphi_1'(\zeta) - (\zeta - z)\varphi_2'(\zeta) - \varphi_3'(\zeta)] \frac{(\zeta - z)\,d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}} \qquad (r > 0,$$

$$(\varkappa = 3 - 4\vartheta)$$

We introduce the notation

$$\varphi(\zeta) = \tfrac{1}{2}[\varphi_1(\zeta) - \varphi_2(\zeta)]$$

$$\psi(\zeta) = \varphi_3'(\zeta) - \frac{\varkappa}{2}[\varphi_1(\zeta) + \varphi_2(\zeta)] + \frac{\zeta}{2}[\varphi_1'(\zeta) + \varphi_2'(\zeta)]$$

where $\phi(\zeta)$ and $\psi(\zeta)$ are holomorphic functions. We represent the equalities (1.6) in the following form:

$$2Gw = \frac{1}{\pi i}\int_{\bar{t}}^{t} [\varkappa\varphi(\zeta) - (2z - \zeta)\varphi'(\zeta) - \psi(\zeta)] \frac{d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}} \tag{1.7}$$

$$2Gu = -\frac{1}{\pi i r}\int_{\bar{t}}^{t} [\varkappa\varphi(\zeta) + (2z - \zeta)\varphi'(\zeta) + \psi(\zeta)] \frac{(\zeta - z)\,d\zeta}{\sqrt{(\zeta - t)(\zeta - \bar{t})}} \qquad (r > 0)$$

For $r = 0$ we have

$$2Gw = \varkappa\varphi(z) - z\varphi'(z) - \psi(z), \qquad u = 0 \tag{1.8}$$

Equations (1.7) may be used in solving the axisymmetric problem for prescribed displacements w_0, u_0 on the boundary of the region. For this the point t should be considered to lie on the contour of the region and the integration is carried out along an arc of the contour. Then the equalities (1.7) reduce to a system of two integral equations for the functions ϕ and ψ.

2. Using the well-known formulas relating the stresses and displacements in cylindrical coordinates, we find the expressions for the stresses

$$\sigma_z = \frac{1}{\pi i}\int_{\bar t}^{t} [2\varphi'(\zeta) - (2z-\zeta)\varphi''(\zeta) - \psi'(\zeta)] \frac{d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}} \tag{1.9}$$

$$\sigma_\theta = \frac{4\nu}{\pi i}\int_{\bar t}^{t} \varphi'(\zeta)\frac{d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}} -$$

$$-\frac{1}{\pi i r^2}\int_{\bar t}^{t} [\varkappa\varphi(\zeta) + (2z-\zeta)\varphi'(\zeta) + \psi(\zeta)] \frac{(\zeta-z)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}}$$

$$\sigma_r = \frac{4(1+\nu)}{\pi i}\int_{\bar t}^{t} \varphi'(\zeta)\frac{d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}} - \sigma_z - \sigma_\theta$$

$$\tau_{rz} = -\frac{1}{\pi i r}\int_{\bar t}^{t} [(2z-\zeta)\varphi''(\zeta) + \psi'(\zeta)] \frac{(\zeta-z)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}} \qquad (r>0)$$

For $r = 0$ we obtain

$$\sigma_z = 2\varphi'(z) - z\varphi''(z) - \psi'(z), \qquad \tau_{rz} = 0$$

$$\sigma_\theta = \sigma_r = (2\nu+1)\varphi'(z) + \frac{1}{2}[z\varphi''(z) + \psi'(z)] \tag{1.10}$$

Expressions (1.7) and (1.9) agree in essence with the formulas of [1-3]. The difference consists only in a different disposition of the real and imaginary axes and some variations in notation.

We will now write out the expressions for the stress resultants p_z and p_r acting on the contour. If a is the angle between the normal to the contour (Fig. 1) and the z-axis, then the tractions p_z and p_r can be expressed in terms of the stresses as follows:

$$p_z = \sigma_z\cos\alpha + \tau_{zr}\sin\alpha, \qquad p_r = \tau_{rz}\cos\alpha + \sigma_z\sin\alpha \tag{1.11}$$

We substitute (1.9) into (1.11); noting that $\cos a = dr/ds$, $\sin a = -dz/ds$, we obtain

$$p_z = -\frac{1}{\pi i r}\frac{d}{ds}\int_{\bar t}^{t} [\varphi(\zeta) - (2z-\zeta)\varphi'(\zeta) - \psi(\zeta)] \frac{(\zeta-z)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar t)}}$$

$$p_r = \frac{1}{\pi i r^2}\frac{d}{ds}\int_{\bar t}^{t} [\varphi(\zeta) + (2z-\zeta)\varphi'(\zeta) + \psi(\zeta)] \left[\frac{(\zeta-z)^2}{\sqrt{(\zeta-t)(\zeta-\bar t)}} + \right.$$

$$+\sqrt{(\zeta-t)(\zeta-\bar{t})}\Big]\,d\zeta-\frac{\sin\alpha}{\pi i r^2}\int_{\bar{t}}^{t}[(3+4\nu)\,\varphi(\zeta)+$$
$$+(2z-\zeta)\,\varphi'(\zeta)+\psi(\zeta)]\,\frac{(\zeta-z)\,d\zeta}{\sqrt{(\zeta-t)(\zeta-\bar{t})}}\qquad(r>0)\qquad(1.12)$$

On the axis of symmetry

$$p_z=2\varphi'(z)-z\varphi''(z)-\psi'(z),\qquad p_r=0\qquad(1.13)$$

In these formulas t is the affix of a contour point. The equalities (1.12) may be used for the solution of the axisymmetric problem, when the external tractions are specified on the boundary.

3. Up to now it has been assumed that the region occupied by the body is finite. But the same reasoning may be applied also to an infinite elastic region with an axisymmetric cavity, only now the branch cut for the radical $\sqrt{(\zeta-t)(\zeta-\bar{t})}$ should be passed through the cavity. All of the formulas obtained previously retain their meanings, with the exception of (1.12). In the equalities (1.12) the directions of p_z and p_r should be reversed, and α should now mean the angle between the z-axis and the inner directed normal. A necessary and sufficient condition for the convergence of the integrals in (1.3) and in the consequent formulas is that the functions $\phi_n(\zeta)$ be holomorphic outside the cavity, including the point at infinity. Moreover, the following equalities must be satisfied:

$$\lim\varphi_1(\zeta)=0,\qquad\lim\zeta\varphi_2(\zeta)=0,\qquad\lim\varphi_3(\zeta)=0\qquad\text{for }\zeta\to\infty\quad(1.14)$$

The functions ϕ and ψ also must be holomorphic outside the cavity and in the neighborhood of infinity have series expansions

$$\varphi(\zeta)=\frac{a_1}{\zeta}+\frac{a_2}{\zeta^2}+\cdots,\qquad\psi(\zeta)=\frac{a_1'}{\zeta}+\frac{a_2'}{\zeta^2}+\cdots\qquad(1.15)$$

where between a_1 and a_1' there is the relation

$$(\varkappa+1)\,a_1+a_1'=0\qquad(1.16)$$

These coefficients may be found easily with the aid of the first of the Formulas (1.12). If Z_0 is the resultant of the external loads applied within the cavity, then the coefficients a_1 and a_1' are equal to

$$a_1=\frac{Z_0}{4\pi(\varkappa+1)},\qquad a_1'=-\frac{Z_0}{4\pi}\qquad(1.17)$$

The displacements and stresses at points lying on the axis of symmetry beneath the cavity (Fig. 1) are determined by Formulas (1.8), (1.10), (1.13). For points lying on the axis of symmetry above the cavity, the signs of these formulas should be reversed.

The formulas obtained in this manner differ from the formulas for an infinite medium with a cavity obtained in [1-3] in that the functions ϕ and ψ appearing in the present work are the derivatives of the corresponding functions in [1-3].

2. Solution for a sphere and a space with a spherical cavity. 1. Let an elastic sphere of radius ρ be subjected to axisymmetric tractions with components p_z and p_r. A section of the sphere formed by a plane passing through the axis of symmetry z is shown in Fig. 2.

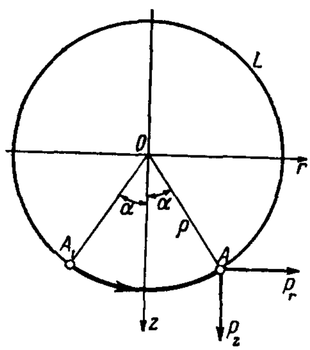

Fig. 2.

We will use Formulas (1.12). For the path of integration we will take the arc A_1A, i.e. we set $\zeta = \sigma$, where σ is the affix of a point on the contour lying between A_1 and A.

We carry out the integrations on the right-hand sides of the equalities (1.12) by parts, noting that

$$t = \rho e^{i\alpha}, \quad \bar{t} = \rho e^{-i\alpha}, \quad z = \rho\cos\alpha, \quad r = \rho\sin\alpha$$

$$ds = \rho\, d\alpha, \qquad \sigma = \rho e^{i\theta} \qquad (-\alpha \leqslant \theta \leqslant \alpha)$$

Upon differentiating the result with respect to α, we obtain

$$p_z = \frac{1}{\pi}\int_{-\alpha}^{+\alpha}[2\varphi'(\sigma) - \bar{\sigma}\varphi''(\sigma) - \psi'(\sigma)]\frac{e^{3i\theta/2}d\theta}{\sqrt{2(\cos\theta-\cos\alpha)}} +$$

$$+\frac{3}{\pi}\rho\int_{-\alpha}^{+\alpha}\varphi''(\sigma)\sqrt{2(\cos\theta-\cos\alpha)}\,e^{3i\theta/2}\,d\theta \tag{2.1}$$

$$p_r = \frac{1}{\pi}\rho\int_{-\alpha}^{+\alpha}[-\varphi''(\sigma)+\bar{\sigma}\varphi'''(\sigma)+\psi''(\sigma)]\sqrt{2(\cos\theta-\cos\alpha)}\,e^{5i\theta/2}d\theta -$$

$$-\frac{5\rho^2}{3\pi}\int_{-\alpha}^{+\alpha}\varphi'''(\sigma)[2(\cos\theta-\cos\alpha)]^{3/2}e^{5i\theta/2}d\theta +$$

$$+\frac{1}{\pi}\int_{-\alpha}^{+\alpha}[(2+4\nu)\varphi'(\sigma)+\bar{\sigma}\varphi''(\sigma)+\psi'(\sigma)]\sqrt{2(\cos\theta-\cos\alpha)}\,e^{3i\theta/2}\,d\theta -$$

$$-\frac{1}{\pi}\rho\int_{-\alpha}^{+\alpha}\varphi''(\sigma)[2(\cos\theta-\cos\alpha)]^{3/2}e^{3i\theta/2}d\theta$$

We introduce holomorphic functions $F(\zeta)$ and $F_1(\zeta)$, whose relation to the functions $\phi(\zeta)$ and $\psi(\zeta)$ is expressed by

$$\varphi'(\zeta) = 2\zeta F'(\zeta) + F(\zeta) \tag{2.2}$$

$$\psi'(\zeta) = 2F(\zeta) + \zeta F'(\zeta) - 2\sigma^2 F''(\zeta) - F_1(\zeta)$$

Then the equalities (2.1) may be put into the form

$$p_z = \frac{1}{\pi}\int_{-\alpha}^{+\alpha} F_1(\sigma)\frac{e^{3i\theta/2}d\theta}{\sqrt{2(\cos\theta-\cos\alpha)}} \tag{2.3}$$

$$p_r \sin\alpha = \frac{1}{\pi}\int_{-\alpha}^{+\alpha}[4(1+\nu)\varphi'(\sigma) + 4\sigma^2 F''(\sigma) - \tag{2.4}$$

$$-\sigma F_1'(\sigma) - F_1(\sigma)]\sqrt{2(\cos\theta-\cos\alpha)}\,e^{3i\theta/2}d\theta$$

Making use of the properties (1.4), which the functions $F(\zeta)$ and $F_1(\zeta)$ also obviously possess, we represent (2.3) in the following form:

$$p_z = \frac{2}{\pi}\operatorname{Re}\int_0^{\alpha} F_1(\sigma)\frac{e^{3i\theta/2}d\theta}{\sqrt{2(\cos\theta-\cos\alpha)}}$$

We multiply this equality by

$$\frac{\sin\alpha\,d\alpha}{\sqrt{2(\cos\alpha-\cos\gamma)}}$$

and integrate between the limits 0 and γ.

On the right-hand side we interchange the order of integration and

$$\int_{\theta}^{\gamma}\frac{\sin\alpha\,d\alpha}{\sqrt{2(\cos\theta-\cos\alpha)}\sqrt{2(\cos\alpha-\cos\gamma)}} = \frac{\pi}{2}.$$

As a result we have

$$\int_0^{\gamma}\frac{p_z\sin\alpha\,d\alpha}{\sqrt{2(\cos\alpha-\cos\gamma)}} = \operatorname{Re}\int_0^{\gamma} F_1(\sigma)\,e^{3i\theta/2}d\theta$$

Whence

$$2\rho e^{-i\gamma/2}\frac{d}{d\gamma}\int_0^{\gamma}\frac{p_z\sin\alpha\,d\alpha}{\sqrt{2(\cos\alpha-\cos\gamma)}} = 2\rho e^{-i\gamma/2}\operatorname{Re}[F_1(\tau)\,e^{3i\gamma/2}] =$$

$$= \tau F_1(\tau) + \frac{\bar{\tau}^2}{\rho}\overline{F_1(\tau)} \qquad (\tau = \rho e^{i\gamma}) \tag{2.5}$$

We multiply the left and right-hand sides of (2.5) by the quantity

$$\frac{1}{2\pi i}\,\frac{d\tau}{\tau-\zeta}$$

and integrate around the closed contour L. In view of well-known properties of Cauchy integrals we obtain

$$F_1(\zeta) = \frac{1}{2\pi i}\int\limits_{(L)} \frac{2\rho}{\zeta}\,\frac{e^{-i\gamma/2}d\tau}{\tau-\zeta}\,\frac{d}{d\gamma}\int\limits_0^\gamma \frac{p_z \sin\alpha\, d\alpha}{\sqrt{2(\cos\alpha-\cos\gamma)}} \tag{2.6}$$

By exactly similar reasoning we find

$$V(\gamma) = 2e^{-3i\gamma/2}\frac{d}{d\gamma}\left[\frac{1}{\sin\gamma}\,\frac{d}{d\gamma}\int\limits_0^\gamma \frac{p_r\sin^2\alpha\, d\alpha}{\sqrt{2(\cos\alpha-\cos\gamma)}}\right]-$$

$$-2ie^{-i\gamma}\frac{d}{d\gamma}\left[e^{-i\gamma/2}\frac{d}{d\gamma}\int\limits_0^\gamma \frac{p_z\sin\alpha\, d\alpha}{\sqrt{2(\cos\alpha-\cos\gamma)}}\right] = 4(1+\nu)\,\varphi'(\tau) + 4\tau^2F''(\tau) -$$

$$-\frac{\bar\tau^2}{\rho^3}\,[\overline{\tau^2F_1(\tau)}]' + \frac{\bar\tau^3}{\rho^3}\,[4(1+\nu)\,\overline{\varphi'(\tau)} + 4\bar\tau^2\overline{F''(\tau)}] - \frac{\bar\tau^3}{\rho^3}\,[\bar\tau\,\overline{F_1(\tau)}] \tag{2.7}$$

and also

$$4v(\zeta) = \frac{1}{2\pi i}\int\limits_{(L)} \frac{V(\gamma)\,d\tau}{\tau-\zeta} = 4(1+\nu)\,\varphi'(\zeta) + 4\zeta^2F''(\zeta) =$$

$$= 4\,[\zeta^2F''(\zeta) + 2(1+\nu)\,\zeta F'(\zeta) + (1+\nu)\,F(\zeta)] \tag{2.8}$$

Solving the differential equation for $F(\zeta)$, we have

$$F(\zeta) = \frac{\zeta^{k_1}}{k_1-k_2}\int v(\zeta)\,\frac{d\zeta}{\zeta^{k_1+1}} - \frac{\zeta^{k_2}}{k_1-k_2}\int v(\zeta)\,\frac{d\zeta}{\zeta^{k_2+1}} \tag{2.9}$$

where k_1 and k_2 are the roots of the characteristic equation

$$k^2 + (1-2\nu)\,k + 1 + \nu = 0$$

The integration constants here are chosen in such a manner that the function $F(\zeta)$ will be holomorphic.

All of the above considerations remain valid also in the case of an elastic space containing a spherical cavity, except the signs are reversed on the right hand sides of the Formulas (2.6), (2.8), (2.9), and the positive directions of p_z and p_r are changed.

2. If the displacements w_0, u_0 are given on the boundary of the sphere, then the expressions with which we start are the equalities (1.7).

Calculations analogous to those carried out above lead to the formulas

$$f_1(\zeta) = \frac{1}{2\pi i}\int\limits_{(L)} \frac{2e^{-i\gamma/2}d\tau}{\tau-\zeta}\,\frac{d}{d\gamma}\int\limits_0^\gamma \frac{2Gw_0\sin\alpha\, d\alpha}{\sqrt{2(\cos\alpha-\cos\gamma)}}$$

$$f(\zeta) = \frac{\zeta^k}{k}\int x(\zeta)\,\frac{d\zeta}{\zeta^k} - \frac{1}{k}\int x(\zeta)\,d\zeta \tag{2.10}$$

where

$$k = \frac{\varkappa + 1}{2\varkappa}, \qquad x(\zeta) = \frac{1}{2\pi i}\int_{(L)} \frac{X(\gamma)}{4\varkappa}\frac{d\tau}{\tau - \zeta}$$

$$X(\gamma) = \frac{2e^{-3i\gamma/i}}{\rho}\frac{d}{d\gamma}\left[\frac{1}{\sin\gamma}\frac{d}{d\gamma}\int_0^\gamma \frac{2Gu_0 \sin^2\alpha\, d\alpha}{\sqrt{2(\cos\alpha - \cos\gamma)}}\right] -$$

$$-\frac{2ie^{-i\gamma}}{\rho}\frac{d}{d\gamma}\left[e^{-i\gamma/2}\frac{d}{d\gamma}\int_0^\gamma \frac{2Gw_0 \sin\alpha\, d\alpha}{\sqrt{2(\cos\alpha - \cos\gamma)}}\right]$$

The relations between the functions $f(\zeta)$ and $f_1(\zeta)$ and the functions ϕ and ψ are

$$\varphi(\zeta) = 2\zeta f'(\zeta) - f(\zeta) \tag{2.11}$$

$$\psi(\zeta) = -\varkappa f(\zeta) + (2\varkappa - 1)\zeta f'(\zeta) - 2\rho^2 f''(\zeta) - f_1(\zeta)$$

3. *Example 1. Elastic sphere of unit radius under a uniform pressure p.* In this case $p_z = -p\cos\alpha$, $p_r = -p\sin\alpha$

$$\int_0^\gamma \frac{p_z \sin\alpha d\alpha}{\sqrt{2(\cos\alpha - \cos\gamma)}} = -p\left(2\sin\frac{\gamma}{2} - \frac{8}{3}\sin^3\frac{\gamma}{2}\right)$$

$$\int_0^\gamma \frac{p_2 \sin^2\alpha d\alpha}{\sqrt{2(\cos\alpha - \cos\gamma)}} = -p\left(\frac{16}{3}\sin^3\frac{\gamma}{2} - \frac{64}{15}\sin^5\frac{\gamma}{2}\right)$$

According to Formulas (2.6) to (2.9) we obtain

$$F_1(\zeta) = -p, \qquad V(\gamma) = -3p, \qquad v(\zeta) = -\tfrac{3}{4}p$$

$$F(\zeta) = -\frac{3p}{4k_1k_2} = -\frac{3p}{4(1+\nu)}$$

From Formula (2.2) we have

$$\varphi'(\zeta) = -\frac{3p}{4(1+\nu)}, \qquad \varphi'(\zeta) = -\frac{1-2\nu}{2(1+\nu)}p$$

Hence the following expressions are correct up to real additive constants

$$\varphi(\zeta) = -\frac{3p}{4(1+\nu)}\zeta, \qquad \psi(\zeta) = -\frac{1-2\nu}{2(1+\nu)}p\zeta$$

According to Formulas (1.10) the stresses on the axis of symmetry are equal to

$$\sigma_z = 2\varphi'(z) - z\varphi''(z) - \psi'(z) = -p, \qquad \tau_{rz} = 0$$

$$\sigma_r = \sigma_\theta = (2\nu + 1)\varphi'(z) + \tfrac{1}{2}[z\varphi''(z) + \psi'(z)] = -p$$

Example 2. Spherical cavity of radius ρ under the action of a uniform pressure p. In this case $p_z = -p\cos\alpha$, $p_r = -p\sin\alpha$. We obtain

$$F_1(\zeta) = -p\frac{\rho^3}{\zeta^3}, \qquad V(\gamma) = -3p, \qquad v(\zeta) = 0, \qquad F(\zeta) = 0$$

$$\varphi'(\zeta) = 0', \qquad \varphi'(\zeta) = p\frac{\rho^3}{\zeta^3}, \qquad \varphi(\zeta) = 0, \qquad \psi(\zeta) = -p\frac{\rho^3}{2\zeta^2}$$

The corresponding stresses on the axis of symmetry are

$$\sigma_z = -p\frac{\rho^3}{z^3}, \qquad \sigma_r = \sigma_\theta = p\frac{\rho^3}{2z^3} \qquad \text{for } z > 0$$

$$\sigma_z = p\frac{\rho^3}{z^3}, \qquad \sigma_r = \sigma_\theta = -p\frac{\rho^3}{2z^3} \qquad \text{for } z < 0$$

Example 3. Elastic space with an absolutely rigid spherical inclusion, which is under the action of an axial force Z_0. Here w_0 = const, $u_0 = 0$

$$X(\gamma) = -2Gw_0\frac{e^{-2i\gamma}}{\rho}, \qquad x(\zeta) = -\frac{2Gw_0\rho}{4\varkappa\zeta^2}$$

$$f_1(\zeta) = 2Gw_0\rho\frac{1}{\xi}, \qquad f(\zeta) = -\frac{2Gw_0\rho}{2(3\varkappa+1)}\frac{1}{\zeta}$$

Whence

$$\varphi(\zeta) = \frac{3Gw_0\rho}{3\varkappa+1}\frac{1}{\zeta}, \qquad \psi(\zeta) = -\frac{3(\varkappa+1)Gw_0\rho}{3\varkappa+1}\frac{1}{\zeta} + \frac{4Gw_0\rho^3}{3\varkappa+1}\frac{1}{\zeta^3}$$

From (1.17) we can write

$$\frac{1}{4\pi}Z_0 = \frac{3(\varkappa+1)Gw_0\rho}{3\varkappa+1}$$

Whence

$$\varphi(\zeta) = \frac{Z_0}{4\pi(\varkappa+1)}\frac{1}{\zeta}, \qquad \psi(\zeta) = -\frac{Z_0}{4\pi}\frac{1}{\zeta} + \frac{Z_0\rho^2}{3\pi(\varkappa+1)}\frac{1}{\zeta^3}$$

For $\rho \to 0$ we obtain the solution for a concentrated force Z_0 applied at the origin

$$\varphi(\zeta) = \frac{Z_0}{4\pi(\varkappa+1)}\frac{1}{\zeta}, \qquad \varphi(\zeta) = -\frac{Z_0}{4\pi}\frac{1}{\zeta}$$

The stresses on the axis of symmetry will be ($z > 0$)

$$\sigma_z = -\frac{(2-\nu)Z_0}{4\pi(1-\nu)}\frac{1}{z^2}, \qquad \sigma_r = \sigma_\theta = \frac{(1-2\nu)Z_0}{8\pi(1-\nu)}\frac{1}{z^2}$$

which coincide with well-known formulas.

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